the section $\xi = 1$ ($\xi = b_0 z$) for different heating times τ_0 have been constructed for a steel half-space by means of (4.4), (4.5).

It is seen from Fig. 1 that the maximum stress diminishes rapidly as τ_0 increases, and for $\tau_0 = 2$ this maximum is around 43% of its value at $\tau_0 = 0$ (instantaneous heating). Thus, the maximum dynamic stress is reduced 57% for a 2 sec heating duration. This indicates that taking account of the finite velocity of heat propagation, the rise in stress due to dynamic effects generally has no practical value.

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ON THE EXISTENCE OF A FIELD OF STRESS RATES IN A HARDENING ELASTIC-PLASTIC MEDIUM

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The boundary value problem for the stress rates and rates of change fields in the quasi-static motion of a volume V of an elastic-plastic medium [1] consists of finding the pairs $\sigma_{ij}^{\cdot}$, $\varepsilon_{ij}^{\cdot}$ related by the governing equations of an appropriate model; here the $\sigma_{ij}^{\cdot}$ should be statically admissible, i.e. should satisfy the equations and boundary conditions

$$\dot{\sigma}_{ij,\,j} = -\dot{X}_i, \qquad \dot{\sigma}_{ij} n_j |_{S_p} = \dot{p}_i \tag{0.1}$$

and $\varepsilon_{ij}^{\cdot}$ should be kinematically admissible, i.e. $2\varepsilon_{ij}^{\cdot} = v_{i,\,j} + v_{j,\,i}$, where

$$v_i |_{S_u} = \dot{u}_{i0} \tag{0.2}$$

Here S_p and S_u are nonintersecting parts of the boundary of the volume V, $X_i^{\cdot}$, $p_i^{\cdot}$, $u_{i0}^{\cdot}$ are specified functions. The question of the existence of a solution of this problem reduces to the question of the functional

$$I(\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot \circ}) = \int_V \left[\frac{1}{2} \varepsilon_{ij}^{\cdot} (\sigma_{kl}^{\cdot *}) \sigma_{ij}^{\cdot *} + \frac{1}{2} \sigma_{ij}^{\cdot} (\varepsilon_{kl}^{\cdot \circ}) \varepsilon_{ij}^{\cdot \circ} - \sigma_{ij}^{\cdot *} \varepsilon_{ij}^{\cdot \circ} \right] dV \quad (0.3)$$

reaching the lower bound in a set of kinematically admissible $\varepsilon_{ij}^{\cdot\circ}$ and statically admissible $\sigma_{ij}^{\cdot *}$. However, its lower bound may not be reached if in the minimization we limit ourselves only to smooth fields.

It is proposed to augment the set of admissible fields $\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot\circ}$ by closing them in the norm L_2 (for v_i° this corresponds to closure in the norm H^1). Some properties of the functional $I(\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot\circ})$ are considered in the augmented set of admissible fields. It is shown that the equivalence of the two problems is conserved, where $I(\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot\circ})$ can be minimized in $\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot\circ}$ or in $\sigma_{ij}^{\cdot *}, \varepsilon_{ij}^{\cdot\circ}$. The lower bound is reached in each of three cases, at a single point. From the fact that v_i° belongs to the Sobolev space $W_2^{(1)}$, there results the absence of surfaces of velocity discontinuity.

Variational principles have been used in plasticity theory to construct models [2] and to investigate the existence and properties of solutions [1, 3].

1. Let us consider a set of kinematically admissible fields $\varepsilon^{\cdot\circ}$ and statistically admissible fields $\sigma^{\cdot *}$. We select some particular velocity field v_p satisfying (0.2) and a particular solution s_p of the system (0.1) (the solution of the linear elasticity theory problem obtained upon appending Hooke's law to (0.1) can be taken as the latter, for example).

Let V_c° be the space of differentiable velocity fields v_c° satisfying the homogeneous boundary condition (0.2), E_c° the space of corresponding fields $(e_c^\circ)_{ij} = {}^1/_2 [(v_c^\circ)_{ij} + (v_c^\circ)_{ji}]$; S_c^* the space of differentiable statically admissible fields s corresponding to the homogeneous system (0.1). Evidently $\sigma^{\cdot *} \in S_c^* + s_p$ and $\varepsilon^{\cdot\circ} \in E_c^\circ + e_p$ are statically and kinematically admissible, respectively.

The functional (0.3) with a lower bound in the set $(S_c^* + s_p)$ $(E_c^\circ + e_p)$ [1] may not reach its lower bound on it. In order to assure achievement of the lower bound of the functional (0.3), let us augment this set as follows. Let S^* and E° be the closure of S_c^* and E_c° in $L_2(V)$, respectively, where $L_2 = L_2(V)$ is understood to be the space of sets of the functions $y = \{y_{ij}(x)\}$ and $y_{ij} = y_{ji}$, defined in V with summable $y_{ij}\, y_{ji}$. The scalar product

$$(y, z)_{L_2} = \int_V (y, z)\, dV, \qquad (y, z) = y_{ij}(x)\, z_{ij}(x)$$

is defined in L_2. It is assumed that the domain V occupied by the medium is bounded.

Let us take $E^\circ + e_p = D^\circ$ and $S^* + s_p = \Sigma^*$, respectively, as the sets of kinematically admissible strain rates and statically admissible fields of the stress rates. The sets D°, Σ^* are independent of the selection of the particular solutions e_p, s_p since $e_{p1} - e_{p2} \in E^\circ$ and $s_{p1} - s_{p2} \in S^*$.

The closure of E_c° in the norm L_2 is equivalent to the closure of V_c° in the norm H^1

$$\| v \|_{H^1}^2 = \int_V \left[\sum_{i,j} (v_{i,j})^2 + \sum_i (v_i)^2 \right] dV \quad (1.1)$$

Let V° denote such a closure. According to (1.1), if some sequence $(v_c^\circ)_n$ converges to v° in H^1, then the corresponding sequence $(e_c^\circ)_n$ converges to e° in L_2.

Conversely, any $e^\circ \in E^\circ$ is a strain rate field of some field $v^\circ \in V^\circ$. In fact, let $(e_c^\circ)_n \rightarrow e^\circ$ in L_2. Then from the application of the known inequalities [4, 5]

$$\int_V \sum_i v_i^2 dV \leqslant C_1 \int_V \sum_{i,j} (v_{i,j})^2 \, dV \tag{1.2}$$

$$\int_V \sum_{i,j} (v_{i,j})^2 \, dV \leqslant C_2 \int_V (v_{i,j} + v_{j,i})(v_{i,j} + v_{j,i}) \, dV$$

to the field $(v_c^\circ)_n - (v_c^\circ)_m$ which vanishes on S_u, it follows that the sequence $(v_c)^\circ{}_n$ is fundamental in the complete space H^1 with the norm (1.1). If $(v_c^\circ)_n \rightarrow v^\circ$ in H^1, then e° as the limit of $(e_c^\circ)_n$ coincides with the strain rate field corresponding to v° by virtue of (1.1).

The inequalities (1.2) are satisfied if the domain V is bounded by a piecewise-continuous differentiable surface without cusps. Henceforth, this condition will be assumed satisfied. Moreover, it is assumed that either S_u is a part of this surface with positive measure, or the whole boundary of V is S_p and then the admissible velocity fields should satisfy additional conditions [4, 6] which exclude the displacement of the medium as a solid body

$$\int_V v dV = 0, \qquad \int_V \operatorname{rot} v dV = 0$$

The inequalities (1.2) then remain valid.

Let us show that L_2 decomposes into the direct sum $L_2 = 2\mu E^\circ + S^*$ (μ is a dimensional constant). If $y_c \in L_2$ is a set of differentiable functions, then

$$y_c = 2\mu e_c^{\circ 2} + s_c^*, \; e_c^\circ \in E_c^\circ, \; s_c^* \in s_c^*.$$

Here $2\,(e_c^\circ)_{ij} = u_{i,j} + u_{j,i}$ and the u_i are found [5, 6] from the system of linear elasticity theory equations

$$\mu\,(u_{i,j} + u_{j,i})_{,j} = (y_c)_{ij,j}$$

$$u_i\,|_{S_u} = 0, \qquad \mu\,(u_{i,j} + u_{j,i})\, n_j\,|_{S_P} = (y_c)_{ij}\, n_j$$

For sufficiently smooth y_c the u_i [6] are also smooth, consequently, $s_c^* = y_c - 2\mu e_c^\circ$. Evidently $s_c^* = y_c - 2\mu e_c^\circ$ satisfies the homogeneous system (0.1), and therefore $s_c^* \in S_c^*$.

Furthermore, since the differentiable y_c are everywhere compact in L_2, and E_i° is orthogonal to S_c^*, and the subspaces $E^\circ \supset E_c^\circ$, $S^* \supset S_c^*$ are closed in the complete space L_2, then for any $y \in L_2$, $y = 2\mu e^\circ + s^*$, $e^\circ \in E^\circ$, $s^* \in S^*$. The subspaces E° and S^* are orthogonal since the sets E_c°, S^* which are everywhere compact and are contained therein are orthogonal. Hence

$$L_2 = 2\mu E^\circ + S^*, \; E^\circ \perp S^*$$

The field $\varepsilon^{\cdot\circ}$ ($\sigma^{\cdot *}$) is kinematically (statically) admissible if and only if $\varepsilon^{\cdot\circ} - e_p$ ($\sigma^{\cdot *} - s_p$) is orthogonal to the subspace S^* (E°).

Let us consider the pair $\sigma^{\cdot *} \in \Sigma^*$, $\varepsilon^{\cdot\circ} \in D^\circ$ connected by the relationship [7]

$$\varepsilon^\cdot(\sigma^\cdot) = A\sigma^\cdot + c_1(f)\, c_2\,[(f', \sigma^\cdot)]\, h\,(f', \sigma^\cdot)\, f' \tag{1.3}$$

$$c_1(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}, \qquad c_2(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

in the case of a hardening elastic-plastic medium, the solution of the problem to see the stress velocity and rate of change fields. Here f' denotes the tensor $\partial f / \partial\sigma_{ij}$, the scalar product is the convolution of the corresponding tensors, A is an operator corresponding to a positive-definite quadratic form of the elastic energy $^1/_2\ (\sigma, A\sigma)$; $f(\sigma, \chi) = 0$ is the equation of the loading surface, χ is the hardening parameter, and h is a known function of σ, χ, ε^p.

Let us find the inversion of (1.3) explicitly. If $c_1(f) = 0$, then $\sigma^\cdot(\varepsilon^\cdot) = A^{-1}\varepsilon^\cdot$. If $c_1(f) = 1$, then one of two cases

$$\sigma^\cdot(\varepsilon^\cdot) = \sigma_1^\cdot(\varepsilon^\cdot) = A^{-1}\varepsilon^\cdot, \qquad (f', \sigma_1^\cdot) \leqslant 0 \tag{1.4}$$

or

$$\sigma^\cdot(\varepsilon^\cdot) = \sigma_2^\cdot(\varepsilon^\cdot) = (A + hF)^{-1}\varepsilon^\cdot, \quad (f', \sigma_2^\cdot) \geqslant 0 \quad \left(F_{ijkl} = \frac{\partial f}{\partial\sigma^{ij}}\frac{\partial f}{\partial\sigma^{kl}}\right) \tag{1.5}$$

should be realized. By multiplication it can be seen that

$$(A + hF)^{-1} = A^{-1} - h_1 A^{-1} F A^{-1}, \ h_1 = h\,[1 + h\,(f', A^{-1}f')]^{-1} \tag{1.6}$$

We note that the signs of $(f', \sigma_1^\cdot)$ and $(f', \sigma_2^\cdot)$ agree. Hence, (1.4), (1.5) can be combined into an expression analogous to (1.3)

$$\sigma^\cdot(\varepsilon^\cdot) = A^{-1}\varepsilon^\cdot - c_1(f)\, c_2\,[(g, \varepsilon^\cdot)]\, h_1\cdot(g, \varepsilon^\cdot)\, g, \ g = A^{-1}f' \tag{1.7}$$

2. Let us examine some properties of the functional (0.3). We assume that the $\sigma_{ij}(x)$, $\varepsilon_{ij}(x)$, $\chi(x)$ given in the domain V are such that $h(x)$, $f'(x)$ are bounded functions measurable in V. Then as is seen from (1.3), (1.7), from $\sigma^{\cdot *} \in L_2$ and $\varepsilon^{\cdot\circ} \in L_2$ there follows $\varepsilon^\cdot(\sigma^{\cdot *}) \in L_2$ and $\sigma^\cdot(\varepsilon^{\cdot\circ}) \in L_2$, respectively. Thus, the functional (0.3) is defined on $\Sigma^* \times D^\circ$ and can, taking account of (1.3), (1.7), be represented as

$$I(\sigma^{\cdot *}, \varepsilon^{\cdot\circ}) = I_1(\sigma^{\cdot *}) + I_2(\varepsilon^{\cdot\circ}) - (\sigma^{\cdot *}, \varepsilon^{\cdot\circ})_{L_2} \tag{2.1}$$

$$I_1(\sigma^{\cdot *}) = \frac{1}{2}\int_V (\sigma^{\cdot *}, A\sigma^{\cdot *})\, dV + \frac{1}{2}\int_{V_p} hc_2\,[(f', \sigma^{\cdot *})]\,(f', \sigma^{\cdot *})^2\, dV \tag{2.2}$$

$$I_2(\varepsilon^{\cdot\circ}) = \frac{1}{2}\int_V (\varepsilon^{\cdot\circ}, A^{-1}\varepsilon^{\cdot\circ})\, dV - \frac{1}{2}\int_{V_p} h_1 c_2\,[(g, \varepsilon^{\cdot\circ})]\,(g, \varepsilon^{\cdot\circ})^2\, dV \tag{2.3}$$

Here $V_p \subseteq V$ is the set of points at which $c_1(f) = 1$, i.e. the stresses reach the yield point. We show that the functional (2.1) is continuous. For example, let us consider $I_1(\sigma^{\cdot *})$, the first term in (2.2) can be easily estimated

$$\begin{aligned} \left|\int_V (\sigma_1^{\cdot *}, A\sigma_1^{\cdot *})\, dV - \int_V (\sigma_2^{\cdot *}, A\sigma_2^{\cdot *})\, dV\right| &\leqslant |A| \int_V |\sigma_1^{\cdot *} + \sigma_2^{\cdot *}\| \times \\ \|\sigma_1^{\cdot *} - \sigma_2^{\cdot *}|\, dV &\leqslant |A|\, \|\sigma_1^{\cdot *} + \sigma_2^{\cdot *}\|\, \|\sigma_1^{\cdot *} - \sigma_2^{\cdot *}\| \end{aligned} \tag{2.4}$$

To estimate the second term in (2.2)

$$a = \left|\int_{V_p} hc_2\,[(f', \sigma_1^{\cdot *})]\,(f', \sigma_1^{\cdot *})^2\, dV - \int_{V_p} hc_2\,[(f', \sigma_2^{\cdot *})]\,(f', \sigma_2^{\cdot *})^2\, dV\right|$$

we separate three parts in V_p

$$V_{12} = \{x \in V_p : (f', \sigma_1^{\cdot *}) > 0, \quad (f', \sigma_2^{\cdot *}) > 0\}$$
$$V_1 = \{x \in V_p : (f', \sigma_1^{\cdot *}) > 0, \quad (f', \sigma_2^{\cdot *}) \leqslant 0\}$$
$$V_2 = \{x \in V_p : (f', \sigma_1^{\cdot *}) \leqslant 0, \quad (f', \sigma_2^{\cdot *}) > 0\}$$

Then

$$a \leqslant \Big| \int_{V_{12}} h(f', \sigma_1^{\cdot *} - \sigma_2^{\cdot *})(f', \sigma_1^{\cdot *} + \sigma_2^{\cdot *})\, dV \Big| +$$
$$\int_{V_1} h(f', \sigma_1^{\cdot *})^2\, dV + \int_{V_2} h(f', \sigma_2^{\cdot *})^2\, dV \leqslant$$
$$\Big| \int_{V_{12}} h(f', \sigma_1^{\cdot *} - \sigma_2^{\cdot *})(f', \sigma_1^{\cdot *} + \sigma_2^{\cdot *})\, dV \Big| +$$
$$\int_{V_1} h(f', \sigma_1^{\cdot *} - \sigma_2^{\cdot *})^2\, dV + \int_{V_2} h(f', \sigma_2^{\cdot *} - \sigma_1^{\cdot *})^2\, dV$$

and further analogously to (2.4).

Thus, the functional $I_1(\sigma^{\cdot *})$ is continuous, and the continuity of $I_2(\varepsilon^{\cdot \circ})$ is proved analogously. The last term in (2.1) is continuous by virtue of the continuity of the scalar product, and therefore, the functional $I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ})$ is continuous.

The values of the functional $I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ})$ satisfy the inequality

$$I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ}) \geqslant 0 \qquad (2.5)$$

for $\sigma^{\cdot *} \in S_c^* + s_p$, $\varepsilon^{\cdot \circ} \in E_c^\circ + e_p$ [1]. Since the sets $S_c^* + s_p$, $E_c^\circ + e_p$ are compact everywhere in Σ^*, D°, respectively, and the functional $I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ})$ is continuous, the inequality (2.5) also holds for any $\sigma^{\cdot *} \in \Sigma^*$, $\varepsilon^{\cdot \circ} \in D^\circ$.

It can be shown that the functional (2.1) is strictly convex, i.e.

$$I(\alpha\sigma_1^{\cdot *} + \beta\sigma_2^{\cdot *}, \alpha\varepsilon_1^{\cdot \circ} + \beta\varepsilon_2^{\cdot \circ}) < \alpha I(\sigma_1^{\cdot *}, \varepsilon_1^{\cdot \circ}) + \beta I(\sigma_2^{\cdot *}, \varepsilon_2^{\cdot \circ}) \qquad (2.6)$$
$$0 < \alpha < 1, \quad \alpha + \beta = 1, \quad \sigma_1^{\cdot *}, \sigma_2^{\cdot *} \in \Sigma^*, \quad \varepsilon_1^{\cdot \circ}, \varepsilon_2^{\cdot \circ} \in D^\circ$$

The functional $I_1(\sigma^{\cdot *})$ is strictly convex since the first integrand in (2.2) is a positive-definite quadratic form, and the second one is

$$\varphi(\sigma^{\cdot *}) = hc_2[(f', \sigma^{\cdot *})]\ (f', \sigma^{\cdot *})^2 = h\, {}^1\!/_2 [\,|(f', \sigma^{\cdot *})| + (f', \sigma^{\cdot *})](f', \sigma^{\cdot *})$$

is a convex function.

Furthermore, let us represent the functional (2.3) as

$$2I_2(\varepsilon^{\cdot \circ}) = \int_{V \setminus V_p} (\varepsilon^{\cdot \circ}, A^{-1}\varepsilon^{\cdot \circ})\, dV + \int_{V_p} (\varepsilon^{\cdot \circ}, B\varepsilon^{\cdot \circ})\, dV + \int_{V_p} h_1 c_2 [(-g, \varepsilon^{\cdot \circ})](g, \varepsilon^{\cdot \circ})^2\, dV$$

where it has been taken into account that

$$1 - c_2[(g, \varepsilon^{\cdot})] = c_2[(-g, \varepsilon^{\cdot})]$$

and the form

$$(\varepsilon^{\cdot}, B\varepsilon^{\cdot}) = (\varepsilon^{\cdot}, A^{-1}\varepsilon^{\cdot}) - h_1(g, \varepsilon^{\cdot})^2 = [1 + h(f', A^{-1}f')]^{-1} \times$$
$$[(\varepsilon^{\cdot}, A^{-1}\varepsilon^{\cdot}) + h(\varepsilon^{\cdot}, A^{-1}\varepsilon^{\cdot})(f', A^{-1}f') - h(\varepsilon^{\cdot}, A^{-1}f')^2]$$

is denoted in terms of B The form B is positive-definite, hence, we find as above that $I_2(\varepsilon^{\cdot \circ})$ is a strictly convex functional.

Finally, for the last term in (2.1) we have

$$-(\alpha\sigma_1^{\cdot *} + \beta\sigma_2^{\cdot *}, \alpha\varepsilon_1^{\cdot \circ} + \beta\varepsilon_2^{\cdot \circ})_{L_2} + \alpha(\sigma_1^{\cdot *}, \varepsilon_1^{\cdot \circ})_{L_2} + \beta(\sigma_2^{\cdot *}, \varepsilon_2^{\cdot \circ})_{L_2} =$$
$$\alpha\beta(\sigma_1^{\cdot *} - \sigma_2^{\cdot *}, \varepsilon_1^{\cdot \circ} - \varepsilon_2^{\cdot \circ})_{L_2} = 0$$

since $\sigma_1^{\cdot *} - \sigma_2^{\cdot *}$ and $\varepsilon_1^{\cdot \circ} - \varepsilon_2^{\cdot \circ}$ belong to orthogonal subspaces. Thus, the functional (2.1) is strictly convex.

Each element $\sigma^{\cdot *} \in \Sigma^*$ and $\varepsilon^{\cdot \circ} \in D^\circ$ can be represented as $\sigma^{\cdot *} = s^* + s_p$, $\varepsilon^{\cdot\circ} = e^\circ + e_p$, $s^* \in S^*$, $e^\circ \in E^\circ$. Hence, the functional (2.1) can be represented as the functional

$$\tilde{I}(s^*, e^\circ) = I(\sigma^{\cdot *}, \varepsilon^{\cdot\circ})$$

defined in the linear space $S^* \times E^\circ$.

The functional $\tilde{I}(s^*, e^\circ)$ has a Gâteaux linear differential

$$D\tilde{I}(s^*, e^\circ; H, \eta) = \lim_{t\to 0} \frac{1}{t}[\tilde{I}(s^* + tH, e^\circ + t\eta) - \tilde{I}(s^*, e^\circ)] = \tag{2.7}$$

$$\lim_{t\to 0} \frac{1}{2t} \int_{V_p} h\{c_2[(f', s^* + s + tH)](f', s^* + s + tH)^2 -$$

$$c_2[(f', s^* + s)](f', s^* + s)^2\}\, dV - \lim_{t\to 0} \frac{1}{2t} \int_{V_p} h_1\{c_2[(g, e^\circ + e_p +$$

$$t\eta)](g, e^\circ + e_p + t\eta)^2 - c_2[(g, e^\circ + e_p)](g, e^\circ + e_p)^2\}\, dV +$$

$$\int_V [(s^* + s_p, AH) + (e^\circ + e_p, A^{-1}\eta) - (s^* + s_p, \eta) - (e^\circ + e_p, H)]\, dV$$

$$(H \in S^*, \eta \in E^\circ)$$

We represent the first term in (2.7) as

$$i_1 = \lim_{t\to 0} \frac{1}{2t}\Big[\int_{V\pm(t)} h(f', \sigma^{\cdot *} + tH)^2 - \int_{V\mp(t)} h(f', \sigma^{\cdot *})^2\, dV +$$

$$\int_{V^+(t)} h\{(f', \sigma^{\cdot *} + tH)^2 - (f', \sigma^{\cdot *})^2\}\, dV$$

$$V\pm(t) = \{x \in V_p : (f', \sigma^{\cdot *} + tH) > 0, \quad (f', \sigma^{\cdot *}) \leqslant 0\}$$
$$V\mp(t) = \{x \in V_p : (f', \sigma^{\cdot *} + tH) \leqslant 0, \quad (f', \sigma^{\cdot *}) > 0\}$$
$$V^+(t) = \{x \in V_p : (f', \sigma^{\cdot *} + tH) > 0, \quad (f', \sigma^{\cdot *}) > 0\}$$

The inequalities

$$0 < (f', \sigma^{\cdot *} + tH) \leqslant t(f', H), \; 0 < (f', \sigma^{\cdot *}) \leqslant -t(f', H)$$

are respectively valid at the points $V\pm$ and $V\mp$, and the function $(f', H)^2$ is summable in the domain V, hence

$$i_1 = \lim_{t\to 0} \int_{V^+(t)} h(f', \sigma^{\cdot *})(f', H)\, dV$$

We show that for arbitrary $\varepsilon > 0$, and sufficiently small $|t|$

$$\operatorname{mes}[U^+ \setminus V^+(t)] < \varepsilon \tag{2.8}$$
$$U^+ = \{x \in V_p : (f', \sigma^{\cdot *}) > 0\}, \; V^+(t) \subseteq U^+$$

In fact, by virtue of the boundedness of V there is a $\alpha > 0$ such that

$$\operatorname{mes} U_\alpha^+ < \varepsilon / 2, \qquad U_\alpha^+ = \{x \in V_p : \alpha > (f', \sigma^{\cdot *}) > 0\} \tag{2.9}$$

Let us consider the set

$$Q(t) = (U^+ \setminus U_\alpha^+) \setminus V^+(t) =$$
$$\{x \in V_p : (f', \sigma^{\cdot *}) \geqslant \alpha, \; (f', \sigma^{\cdot *} + tH) \leqslant 0\}$$

The inequality

$$\alpha \leqslant (f', \sigma^{\cdot *}) \leqslant -t(f', H) = |t||(f', H)|$$

is valid in this set, and therefore

$$Q(t) \subseteq \left\{x \in U^+ : |(f', H)| \geqslant \frac{\alpha}{|t|}\right\} \subseteq \left\{x \in U^+ : |(f', H)| \geqslant \frac{\alpha}{\delta}\right\} \equiv M_{\alpha/\delta} \quad (|t| < \delta) \tag{2.10}$$

The function $|(f', H)|$ is summable in the domain V, hence there exists a $\delta > 0$ such that $\operatorname{mes} M_{\alpha/\delta} < \varepsilon / 2$. Then we find from (2.9) and (2.10) that the inequality (2.8) holds for $|t| < \delta$, and consequently,

$$i_1 = \int_{U+} h(f', \sigma^{\cdot *})(f', H)\, dV = \int_{V_p} hc_2[(f', \sigma^{\cdot *})](f', \sigma^{\cdot *})(f', H)\, dV \tag{2.11}$$

The second term in (2.7) can analogously be converted into

$$-\int_{V_p} h_1 c_2 [(g, \varepsilon^{\cdot \circ})](g, \varepsilon^{\cdot \circ})(g, \eta)\, dV \tag{2.12}$$

Substituting (2.11), (2.12) into (2.7), we finally obtain

$$D\widetilde{I}(s^*, e^\circ; H, \eta) = \int_V [(\varepsilon^\cdot(\sigma^{\cdot *}) - \varepsilon^{\cdot \circ}, H) + (\sigma^\cdot(\varepsilon^{\cdot \circ}) - \sigma^{\cdot *}, \eta)]\, dV \tag{2.13}$$

This calculation has been carried out in [1] under the assumption of differentiability of $\sigma^{\cdot *}$, $\varepsilon^{\cdot \circ}$.

Finally, let us show that the functional $\widetilde{I}(s^*, e^\circ)$ is growing, i.e. $\widetilde{I}(s^*, e) \to \infty$ for $\|s^*\| + \|e^\circ\| \to \infty$. In fact, discarding the second member in I_1 and integrating over the whole domain V in the second member of I_2, we obtain

$$\begin{aligned}\widetilde{I}(s^*, e^\circ) \geqslant {}& {}^1/_2(s^*, As^*)_{L_2} + {}^1/_2(e^\circ, Be^\circ)_{L_2} + \\ & {}^1/_2(s_p\, As_p)_{L_2} + {}^1/_2(e_p, Be_p)_{L_2} - (e^\circ, s_p)_{L_2} - (s^*, e_p)_{L_2} - \\ & (s_p, e_p)_{L_2} + (s^*, As_p)_{L_2} + (e^\circ, Be_p)_{L_2}\end{aligned}$$

The forms (s^*, As^*) and (e°, Be°) in the first two members are positive definite, and the remaining terms are linear in s^*, e° (s_p and e_p are fixed), hence $\widetilde{I}(s^*, e^\circ)$ is increasing.

Thus, the functional $\widetilde{I}(s^*, e^\circ)$, defined in the Hilbert space $S^* \times E^\circ \subset L_2 \times L_2$ is continuous therein, strictly convex, increasing, and has a linear Gâteaux differential

$$D\widetilde{I}(s^*, e^\circ; H, \eta) = (\varepsilon^\cdot(\sigma^{\cdot *}) - \varepsilon^{\cdot \circ}, H)_{L_2} + (\sigma^\cdot(\varepsilon^{\cdot \circ}) - \sigma^{\cdot *}, \eta)_{L_2} \tag{2.14}$$

$$(\sigma^{\cdot *} = s^* + s_p,\ \varepsilon^{\cdot \circ} = e^\circ + e_p)$$

3. There results from the listed properties of the functional $\widetilde{I}(s^*, e^\circ)$ [8] that

(a) For every fixed e° from E° there exists an element of the space S^* — $s^{**}[e^\circ]$ on which the lower bound is achieved

$$\widetilde{I}(s^{**}[e^\circ], e^\circ) = \inf_{s^* \in S^*} \widetilde{I}(s^*, e^\circ) = \inf_{\sigma^{\cdot *} \in \Sigma^*} I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ}) = m_1(\varepsilon^{\cdot \circ}) \tag{3.1}$$

(b) For every fixed s^* from S^* there exists an element of the space E° — $e^{\circ\circ}[s^*]$ on which the lower bound is achieved

$$\widetilde{I}(s^*, e^{\circ\circ}[s^*]) = \inf_{e^\circ \in E^\circ} \widetilde{I}(s^*, e^\circ) = \inf_{\varepsilon^{\cdot \circ} \in D^\circ} I(\sigma^{\cdot *}, \varepsilon^{\cdot \circ}) = m_2(\sigma^{\cdot *}) \tag{3.2}$$

(c) There exists an element of the space $S^* \times E^\circ - (s_m{}^*, e_m{}^\circ)$ on which the lower bound is achieved

$$\tilde{I}(s_m{}^*, e_m{}^\circ) = \inf_{s^*\in S^*, e^\circ\in E^\circ} \tilde{I}(s^*, e^\circ) = \inf_{\sigma^{\cdot *}\in\Sigma^*, \varepsilon^{\cdot\circ}\in D^\circ} I(\sigma^{\cdot *}, \varepsilon^{\cdot\circ}) = m \qquad (3.3)$$

In each of the three cases the lower bound is achieved at a single point.

For a Gâteaux differentiable convex functional, the necessary and sufficient condition for reaching the lower bound is that its differential vanishes. Hence, we find from (2.14) in the case (a) ($\eta = 0$)

$$(\varepsilon^\cdot(\sigma^{\cdot **}) - \varepsilon^{\cdot\circ}, H)_{L_2} = 0$$

for any H from S^*, and, consequently, $\varepsilon^\cdot(\sigma^{\cdot **})$ is a kinematically admissible field. The pair $\sigma^{\cdot **}$, $\varepsilon^\cdot(\sigma^{\cdot **})$ is a solution of the elastic-plastic problem. Conversely if a kinematically admissible field $\varepsilon^\cdot(\sigma^{\cdot *})$ corresponds to the field $\sigma^{\cdot *}$ from Σ^*, i.e. $\sigma^{\cdot *}$, $\varepsilon^\cdot(\sigma^{\cdot *})$ is a solution, then

$$\varepsilon^\cdot(\sigma^{\cdot *}) \in D^\circ, \quad \varepsilon^\cdot(\sigma^{\cdot *}) - \varepsilon^{\cdot\circ} \in E^\circ, \quad (\varepsilon^\cdot(\sigma^{\cdot *}) - \varepsilon^{\cdot\circ}, H)_{L_2} = 0$$

and $m_1(\varepsilon^{\cdot\circ})$ is achieved on $\sigma^{\cdot *}$.

Analogous assertions hold in case (b) as well. Finally, it is evident in case (c) that $\sigma_m{}^{\cdot *}$, $\varepsilon_m{}^{\cdot\circ}$ is a solution of the elastic-plastic problem and, conversely, the solution reaches the minimum of the functional $I(\sigma^{\cdot m}, \varepsilon^{\cdot\circ})$. There hence results that it is unique.

Consequently, $m_1(\varepsilon^{\cdot\circ})$ is reached for any $\varepsilon^{\cdot\circ}$ on the same element $\sigma^{\cdot **}$ from Σ^*, and $m_2(\sigma^{\cdot *})$ for any $\sigma^{\cdot *}$ on the same $\varepsilon^{\cdot\circ\circ}$ from D°. Moreover

$$\varepsilon^{\cdot\circ} = \varepsilon^{\cdot\circ\circ} = \varepsilon^\cdot(\sigma^{\cdot **}), \quad \sigma^{\cdot *} = \sigma^{\cdot **} = \sigma^\cdot(\varepsilon^{\cdot\circ\circ})$$

Thus, if finding the pair $\sigma^{\cdot *} \in \Sigma^*$, $\varepsilon^{\cdot\circ} \in D^\circ$ connected by the relationships (1.3), (1.7) is understood to be the solution of the problem of determining the stress velocity and rate of charge fields in an elastic-plastic medium, the following assertion has been proved.

If a domain V bounded by a piecewise-continuous differentiable surface without cusps is filled with a hardening elastic-plastic medium, then for a given distribution $\sigma_{ij}(x)$, $\varepsilon_{ij}{}^p(x)$, $\chi(x)$ such that h and f_{ij}' are bounded functions measurable in V, there exists a unique solution $\sigma_m{}^{\cdot *}$, $\varepsilon_m{}^{\cdot\circ}$. Hence, $\sigma_m{}^{\cdot *}$, and correspondingly $\varepsilon_m{}^{\cdot *}$ and the pair $\sigma_m{}^{\cdot *}$, $\varepsilon_m{}^{\cdot\circ}$ yields the solution of the problem of minimization (3.1) of the functional (0.3) for any $\varepsilon^{\cdot\circ}$, (3.2) for any $\sigma^{\cdot *}$ and (3.3). Conversely, the solution of any of these problems permits determining $\sigma_m{}^{\cdot *}$ or $\varepsilon_m{}^{\cdot\circ}$.

As has been shown above, any field $\varepsilon^{\cdot\circ} \in D^\circ$ is a strain rate field of some velocity field v from the space $H^1(V)$. Thus, the field v has derivatives in the Sobolev sense, and consequently, there are no two-dimensional velocity discontinuity surfaces in the domain V. The reasoning carried out does not permit exclusion of the presence of singularities on manifolds of dimensionality less than two. However, in the case of a one-dimensional problem, $H^1 \subset C$ and the velocity field is continuous according to the Sobolev embedding theorem [4].

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EFFECT OF INTERCONVERTIBILITY OF ELECTROMAGNETIC AND GRAVITATIONAL WAVES IN STRONG EXTERNAL ELECTROMAGNETIC FIELDS AND THE PROPAGATION OF WAVES IN THE FIELD OF A CHARGED "BLACK HOLE"

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It is shown that the behavior of an arbitrary wave propagating in the field of a nonrotating charged black hole is defined (with the use of quadratures) by four functions. Each of these functions obeys its second order equation of the wave kind. Short electromagnetic waves falling onto a black hole are reflected by its field in the form of gravitational and electromagnetic waves whose amplitude was explicitly determined. In the case of the wave carrying rays winding around the limit cycle the reflection and transmission coefficients were obtained in the form of analytic expressions.

Various physical processes taking place inside, as well as outside a collapsing star, may induce perturbations of the gravitational, electromagnetic and other fields, and lead to the appearance in the surrounding space of waves of various kinds which propagate over a distorted background and are dissipated along its inhomogeneities.

In the absence of rotation and charge in a star, the analysis of small perturbations of the gravitational fields is based on the system of Einstein equations linearized around the Schwarzschild solution. In [1, 2] this system of equations, after expansion of perturbations in spherical harmonics and Fourier transformation with respect to time, was reduced to two independent linear ordinary differential equations of second order of the form of the stationary Schrödinger equation for a particle in a potential force field. Each of these equations defines one of two possible independent perturbation kinds: "even" and "odd"